BULLETIN DES
SCIENCES
MATHÉMATIQUES

Bull. Sci. math. 132 (2008) 218–231

www.elsevier.com/locate/bulsci

Hopf bifurcation for degenerate singular points of multiplicity $2n-1$ in dimension 3

Jaume Llibre [a,*,1], Hao Wu [b,2]

[a] *Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain*
[b] *Centre de Recerca Matemàtica, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain*

Received 3 January 2007; accepted 9 January 2007

Available online 12 March 2007

Abstract

The main purpose of this paper is to study the Hopf bifurcation for a class of degenerate singular points of multiplicity $2n-1$ in dimension 3 via averaging theory. More specifically, we consider the system

$$\begin{aligned}
\dot{x} &= -H_y(x,y) + P_{2n}(x,y,z) + \varepsilon P_{2n-1}(x,y),\\
\dot{y} &= H_x(x,y) + Q_{2n}(x,y,z) + \varepsilon Q_{2n-1}(x,y),\\
\dot{z} &= R_{2n}(x,y,z) + \varepsilon c z^{2n-1},
\end{aligned}$$

where

$$\begin{aligned}
H &= \frac{1}{2n}\left(x^{2l}+y^{2l}\right)^m, \quad n = lm,\\
P_{2n-1} &= x\left(p_1 x^{2n-2} + p_2 x^{2n-3} y + \cdots + p_{2n-1} y^{2n-2}\right),\\
Q_{2n-1} &= y\left(p_1 x^{2n-2} + p_2 x^{2n-3} y + \cdots + p_{2n-1} y^{2n-2}\right),
\end{aligned}$$

and P_{2n}, Q_{2n} and R_{2n} are arbitrary analytic functions starting with terms of degree $2n$. We prove using the averaging theory of first order that, moving the parameter ε from $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small, from the origin it can bifurcate $2n-1$ limit cycles, and that using the averaging theory of second order from the origin it can bifurcate $3n-1$ limit cycles when $l = 1$.

* Corresponding author.
E-mail addresses: jllibre@mat.uab.es (J. Llibre), hwu@crm.es (H. Wu).
[1] The author has been supported by the grants MCYT-Spain MTM2005-06098-C02-01 and CIRIT-Spain 2005SGR 00550.
[2] The author has been supported by a CRM grant.

0007-4497/$ – see front matter
doi:10.1016/j.bulsci.2007.01.003

MSC: 37G15; 37D45

Keywords: Limit cycles; Hopf bifurcation; Averaging theory

1. Introduction and statement of the main results

Let

$$\dot{x} = P(x, y, z), \qquad \dot{y} = Q(x, y, z), \qquad \dot{z} = R(x, y, z),$$

be an analytic system in $\mathbb{R}^3$ starting with terms in P, Q and R of order $2n-1$. Then, here we say that the singular point at the origin of $\mathbb{R}^3$ has *multiplicity* $2n-1$.

The main purpose of this paper is to study the Hopf bifurcation which takes place at the singular point located at the origin for a subclass analytic differential equations in $\mathbb{R}^3$ of the form

$$\begin{aligned}
\dot{x} &= \overline{P}_{2n-1}(x, y, z) + \overline{P}_{2n}(x, y, z), \\
\dot{y} &= \overline{Q}_{2n-1}(x, y, z) + \overline{Q}_{2n}(x, y, z), \\
\dot{z} &= \overline{R}_{2n-1}(x, y, z) + \overline{R}_{2n}(x, y, z),
\end{aligned} \tag{1}$$

where $\overline{P}_{2n-1}$, $\overline{Q}_{2n-1}$ and $\overline{R}_{2n-1}$ are homogeneous polynomials of degree $2n-1$, and $\overline{P}_{2n}$, $\overline{Q}_{2n}$ and $\overline{R}_{2n}$ are analytical functions starting with terms of order $2n$.

In general Hopf bifurcation is well studied for singular points which have an eigenvalue of the form $\alpha(\varepsilon) \pm \beta(\varepsilon)i$ with $\alpha(0) = 0$ and $\alpha'(0) \neq 0$. Also in dimension 2 the Hopf bifurcation can be obtained for the singular points having eigenvalues of the form $\pm\beta i$ using the so-called Lyapunov constants, see for instance [1,2]. But for systems (1) with $n > 1$, the singular point located at the origin is degenerated and all its eigenvalues are zero, so the standard techniques for studying the limit cycles that bifurcate from the origin changing a parameter cannot be applied.

We shall study the Hopf bifurcation of a subclass of systems (1) using the averaging theory, see Section 2. Our main results are as follows.

Theorem 1. *We consider the differential systems in* $\mathbb{R}^3$ *given by*

$$\begin{aligned}
\dot{x} &= -H_y(x, y) + P_{2n}(x, y, z) + \varepsilon P_{2n-1}(x, y), \\
\dot{y} &= H_x(x, y) + Q_{2n}(x, y, z) + \varepsilon Q_{2n-1}(x, y), \\
\dot{z} &= R_{2n}(x, y, z) + \varepsilon c z^{2n-1},
\end{aligned} \tag{2}$$

where

$$\begin{aligned}
H &= \frac{1}{2n}\left(x^{2l} + y^{2l}\right)^m, \quad n = lm, \\
P_{2n-1} &= x\left(p_1 x^{2n-2} + p_2 x^{2n-3} y + \cdots + p_{2n-1} y^{2n-2}\right), \\
Q_{2n-1} &= y\left(p_1 x^{2n-2} + p_2 x^{2n-3} y + \cdots + p_{2n-1} y^{2n-2}\right),
\end{aligned}$$

and P_{2n}, Q_{2n} *and* R_{2n} *are arbitrary analytic functions starting with terms of degree* $2n$. *Then, moving the parameter* ε *from* $\varepsilon = 0$ *to* $\varepsilon \neq 0$, *from the origin of system* (2) *it can bifurcate* $2n-1$ *limit cycles.*

Theorem 2. *Under the assumptions of Theorem* 1 *and for the case* $l = 1$, *we have that moving the parameter* ε *from* $\varepsilon = 0$ *to* $\varepsilon \neq 0$, *from the origin of system* (2) *it can bifurcate* $3n - 1$ *limit cycles.*

Theorem 1 will be proved in Section 3 using the averaging theory of first order, and Theorem 2 will be proved in Section 4 using the averaging theory of second order.

We remark that in the proof of Theorem 1 we only use the terms of degree $2n - 1$ and $2n$ of system (2), but in the proof of Theorem 2 we need to use the terms of degree $2n - 1$, $2n$ and $2n + 1$ of system (2).

2. Limit cycles via averaging theory

Under good assumptions we can say that the averaging method [5,6] gives a quantitative relation between the limit cycles of some non-autonomous periodic differential system and the singular points of its averaged differential system, which is an autonomous one. The next theorem provides a first order approximation in ε for the limit cycles of a periodic differential system, for a proof see Theorem 2.6.1 of Sanders and Verhulst [5] and Theorem 11.5 of Verhulst [6].

Theorem 3 *(Averaging method up to first order in* ε*). We consider the following two initial value problems*

$$\dot{x} = \varepsilon f(t, x) + \varepsilon^2 g(t, x, \varepsilon), \quad x(0) = x_0, \tag{3}$$

and

$$\dot{y} = \varepsilon F^0(y), \quad y(0) = x_0, \tag{4}$$

where x, y, $x_0 \in \Omega$ *an open subset of* $\mathbb{R}^n$, $t \in [0, \infty)$, $\varepsilon \in (0, \varepsilon_0]$, f *and* g *are periodic of period* T *in the variable* t, *and* $F^0(y)$ *is the averaged function of* $f(t, x)$ *with respect to* t, *i.e.,*

$$F^0(y) = \frac{1}{T} \int_0^T f(t, y)\, dt. \tag{5}$$

Suppose: (i) f, *its Jacobian* $\partial f / \partial x$, *its Hessian* $\partial^2 f / \partial x^2$, g *and its Jacobian* $\partial g / \partial x$ *are defined, continuous and bounded by a constant independent on* ε *in* $[0, \infty) \times \Omega$ *and* $\varepsilon \in (0, \varepsilon_0]$; (ii) T *is a constant independent of* ε; *and* (iii) $y(t)$ *belongs to* Ω *on the interval of time* $[0, 1/\varepsilon]$. *Then the following statements hold.*

(a) *On the time scale* $1/\varepsilon$ *we have that* $x(t) - y(t) = \mathrm{O}(\varepsilon)$, *as* $\varepsilon \to 0$.

(b) *If* p *is a singular point of the averaged system* (4) *such that the determinant of the Jacobian matrix*

$$\left. \frac{\partial F^0}{\partial y} \right|_{y=p} \tag{6}$$

is not zero, then there exists a limit cycle $\phi(t, \varepsilon)$ *of period* T *for the system* (3) *which is close to* p *and such that* $\phi(t, \varepsilon) \to p$ *as* $\varepsilon \to 0$.

(c) *The stability or instability of the limit cycle* $\phi(t, \varepsilon)$ *is given by the stability or instability of the singular point* p *of the averaged system* (4). *In fact, the singular point* p *has the stability behavior of the Poincaré map associated to the limit cycle* $\phi(t, \varepsilon)$.

In [3] the authors have succeeded to weaken the hypotheses of the averaging method for the existence of periodic orbits using the Brouwer degree theory.

The next theorem provides a second order approximation for the solutions of a periodic differential system, for a proof see Llibre [4] and Theorem 3.5.1 of Sanders and Verhulst [5].

Theorem 4 *(Averaging method up to second order in ε). We consider the following two initial value problems*

$$\dot{x} = \varepsilon f(t,x) + \varepsilon^2 g(t,x) + \varepsilon^3 R(t,x,\varepsilon), \quad x(0) = x_0, \tag{7}$$

and

$$\dot{y} = \varepsilon F^0(y) + \varepsilon^2 F^{10}(y) + \varepsilon^2 G^0(y), \quad y(0) = x_0, \tag{8}$$

with $f, g : [0,\infty) \times D \to \mathbb{R}^n$, $R : [0,\infty) \times D \times (0,\varepsilon_0] \to \mathbb{R}^n$, D an open subset of $\mathbb{R}^n$, f, g and R periodic of period T in the variable t,

$$F^1(t,x) = \frac{\partial f}{\partial x} y^1(t,x) - \frac{\partial y^1}{\partial x} F^0(x), \tag{9}$$

where

$$y^1(t,x) = \int_0^t \big(f(s,x) - F^0(x)\big)\, ds + z(x),$$

with $z(x)$ a C^1 function such that the averaged of y^1 is zero. Of course, F^0, F^{10} and G^0 denote the averaged functions of f, F^1 and g, respectively, defined as in (5). *Suppose*: (i) *$\partial f/\partial x$ is Lipschitz in x, g and R are Lipschitz in x and all these functions are continuous on their domain of definition*; (ii) *$|R(t,x,\varepsilon)|$ is bounded by a constant uniformly in* $[0, L/\varepsilon) \times D \times (0,\varepsilon_0]$; (iii) *$T$ is independent on ε*; (iv) *$y(t)$ belongs to D on the time-scale $\frac{1}{\varepsilon}$. Then the following results hold.*

(a) *On the time-scale $\frac{1}{\varepsilon}$ we have that*

$$x(t) = y(t) + \varepsilon y^1\big(t, y(t)\big) + \mathrm{O}\big(\varepsilon^2\big), \quad \text{as } \varepsilon \to 0.$$

Assume that $F^0(y) \equiv 0$ in addition.

(b) *If p is an equilibrium point of the averaged system* (8) *such that*

$$\frac{\partial}{\partial y}\big(F^{10}(y) + G^0(y)\big)\bigg|_{y=p} \neq 0, \tag{10}$$

then there exists a T-periodic solution $\phi(t,\varepsilon)$ of Eq. (7) *which is close to p such that $\phi(t,\varepsilon) \to p$ as $\varepsilon \to 0$.*

(c) *The stability or instability of the limit cycle $\phi(t,\varepsilon)$ is given by the stability or instability of the singular point p of the averaged system* (8). *In fact, the singular point p has the stability behavior of the Poincaré map associated to the limit cycle $\phi(t,\varepsilon)$.*

3. Proof of Theorem 1

We assume that

$$P_{2n} = \sum_{i+j+k=2n} A_{i,j,k} x^i y^j z^k + \text{H.O.T.},$$
$$Q_{2n} = \sum_{i+j+k=2n} B_{i,j,k} x^i y^j z^k + \text{H.O.T.},$$
$$R_{2n} = \sum_{i+j+k=2n} C_{i,j,k} x^i y^j z^k + \text{H.O.T.},$$

where as usual H.O.T. means higher order terms. By the generalized cylinder transformation

$$x = r Cs\,\theta, \qquad y = r\ Sn\,\theta, \qquad z = z,$$

where $Cs\theta$ and $Cs\theta$ are defined by the solution of the following system

$$\dot{Cs}\theta = -Sn^{2l-1}\theta, \qquad \dot{Sn}\theta = Cs^{2l-1}\theta,$$

with the initial condition $Cs(0) = 1$ and $Sn(0) = 0$. Let $I_{p,q} = \int_0^\theta Sn^p\varphi\ \ Cs^q\varphi\, d\varphi$. Moreover, they satisfy the following properties:

1. $Cs = \cos$ and $Sn = \sin$ if $l = 1$.
2. $Sn^{2l}\theta + Cs^{2l}\theta = 1$.
3. $T = \frac{2}{l} B(1/(2l), 1/(2l))$, where $B(\cdot,\cdot)$ is the Beta function.
4. $I_{2l-1,q} = -\frac{Cs^{q+1}\theta}{q+1}$.
5. $I_{p,2l-1} = \frac{Sn^{p+1}\theta}{p+1}$.
6. $I_{p,q} = \frac{Sn^{p+1}\theta\, Cs^{q-2l+1}\theta}{p+1} + \frac{q-2l+1}{p+1} I_{p+2l,q-2l}$.
7. $I_{p,q} = -\frac{Sn^{p-2l+1}\theta\, Cs^{q+1}\theta}{q+1} + \frac{p-2l+1}{q+1} I_{p-2l,q+2l}$.
8. $\int_0^T Sn^p\varphi\, Cs^q\varphi\, d\varphi = \frac{2}{l} B((p+1)/(2l), (q+1)/(2l))$ if p and q are even.
9. $\int_0^T Sn^p\varphi\, Cs^q\varphi\, d\varphi = 0$ if p or q is odd.

Proposition 5. *Assume $\mathcal{P}_d$ is the homogeneous polynomial function in $\mathbb{R}[x,y]$ of degree d, if d is odd then $\int_0^{2\pi} \mathcal{P}_d(\cos\theta, \sin\theta)\, d\theta = 0$.*

Proof. It is trivial by property 9. □

System (2) in the variables (r,θ,z) becomes

$$\begin{aligned}
\dot r &= \sum_{i=0}^{2n} K_{2n-i,i}(\theta) r^{2n-i} z^i + O_{2n+1}(r,z) + \varepsilon\Gamma(\theta) r^{2n-1},\\
\dot\theta &= \frac{1}{r}\left(r^{2n-1} + \sum_{i=0}^{2n} L_{2n-i,i}(\theta) r^{2n-i} z^i\right) + O_{2n}(r,z),\\
\dot z &= \sum_{i=0}^{2n} M_{2n-i,i}(\theta) r^{2n-i} z^i + O_{2n+1}(r,z) + \varepsilon c z^{2n-1},
\end{aligned} \tag{11}$$

where

$$\Gamma(\theta)=\sum_{i=1}^{2n-1} p_i CVs^{2n-i-1}\theta Sn^{i-1}\theta,$$

$$K_{2n-i,i}(\theta)=\sum_{r+s=2n-i}\left(A_{r,s,i}CVs^{r+2l-1}\theta Sn^{s}\theta+B_{r,s,i}CVs^{r}\theta Sn^{s+2l-1}\theta\right),$$

$$L_{2n-i,i}(\theta)=\sum_{r+s=2n-i}\left(B_{r,s,i}Cs^{r+1}\theta Sn^{s}\theta-A_{r,s,i}Cs^{r}\theta Sn^{s+1}\theta\right),$$

$$M_{2n-i,i}(\theta)=\sum_{r+s=2n-i} C_{r,s,i}CVs^{r}\theta Sn^{s}\theta,\quad \text{in particular } G_{0,2n}=C_{0,0,2n}.$$

By the coordinate transformation $r=\varepsilon R$ and $z=\varepsilon\xi$, system (11) is orbitally equivalent to the following one

$$\begin{aligned}\frac{dR}{d\theta}&=\varepsilon R\frac{\Gamma(\theta)R^{2n-1}+\sum_{i=0}^{2n}K_{2n-i,i}(\theta)r^{2n-i}z^{i}}{R^{2n-1}+\varepsilon\sum_{i=0}^{2n}L_{2n-i,i}(\theta)r^{2n-i}z^{i}}+\mathrm{O}(\varepsilon^{2}),\\ \frac{d\xi}{d\theta}&=\varepsilon R\frac{c\xi^{2n-1}+\sum_{i=0}^{2n}M_{2n-i,i}(\theta)r^{2n-i}z^{i}}{R^{2n-1}+\varepsilon\sum_{i=0}^{2n}L_{2n-i,i}(\theta)r^{2n-i}z^{i}}+\mathrm{O}(\varepsilon^{2}).\end{aligned}\tag{12}$$

Expanding the right part of system (12) with respect to the variable ε, we have

$$\begin{aligned}\frac{dR}{d\theta}&=\varepsilon f_1(\theta,R,\xi)+\varepsilon^2 g_1(\theta,R,\xi,\varepsilon),\\ \frac{d\xi}{d\theta}&=\varepsilon f_2(\theta,R,\xi)+\varepsilon^2 g_2(\theta,R,\xi,\varepsilon),\end{aligned}\tag{13}$$

where

$$f_1(\theta,R,\xi)=\frac{1}{R^{2n-2}}\left(\Gamma(\theta)R^{2n-1}+\sum_{i=0}^{2n}K_{2n-i,i}(\theta)r^{2n-i}z^{i}\right),$$

$$f_2(\theta,R,\xi)=\frac{1}{R^{2n-2}}\left(c\xi^{2n-1}+\sum_{i=0}^{2n}M_{2n-i,i}(\theta)r^{2n-i}z^{i}\right).$$

Furthermore, $\int_0^T K_{2n-p,p}(\theta)\,d\theta=0$, if p is even and $\int_0^T M_{2n-q,q}(\theta)\,d\theta=0$, if q is odd by Proposition 5.

Let Ω be the open subset and ε the positive number which appear in the statement of Theorem 3. Then, it is easy to verify that system (13) satisfies the assumption of Theorem 3 if we take Ω as an open disk centered at the origin $(R,\xi)=(0,0)$ of $\mathbb{R}^2$ and a sufficiently small ε_0. Since

$$F_i^0(R,\xi)=\frac{1}{T}\int_0^T f_i(\theta,R,\xi)\,d\theta,$$

for $i=1$ and 2, then we get that

$$F_1^0(R,\xi)=\frac{1}{R^{2n-2}}\left(\alpha_0R^{2n-1}+\alpha_1R^{2n-1}\xi+\alpha_2R^{2n-3}\xi^3+\cdots+\alpha_nR\xi^{2n-1}\right),$$

$$F_2^0(R,\xi)=\frac{1}{R^{2n-2}}\left(c\xi^{2n-1}+\beta_1R^{2n}+\beta_2R^{2n-2}\xi^2+\cdots+\beta_nR^2\xi^{2n-2}+C_{0,0,2n}\xi^{2n}\right),$$

where

$$\alpha_0 = \frac{1}{T}\int_0^T \Gamma(\theta)\,d\theta = \frac{\sum_{k=1}^{n} p_{2k-1} B\left(\frac{2n-2k+1}{2l}, \frac{2k-1}{2l}\right)}{B\left(\frac{1}{2l}, \frac{1}{2l}\right)},$$

and

$$\begin{aligned}
\alpha_i &= \frac{1}{T}\int_0^T H_{2n-2i+1,2i-1}(\theta)\,d\theta \\
&= \frac{\sum_{k=0}^{n-i} B\left(\frac{2(n+l-i-k)+1}{2l}, \frac{2k+1}{2l}\right)(A_{2(n+k-i)+1,2k,2i-1} + B_{2k,2(n+k-i)+1,2i-1})}{B\left(\frac{1}{2l}, \frac{1}{2l}\right)}, \\
\beta_i &= \frac{1}{T}\int_0^T G_{2(n-i+1),2(i-1)}(\theta)\,d\theta \\
&= \frac{\sum_{k=1}^{n-i+1} B\left(\frac{2(n-k-i)-1}{2l}, \frac{2k+1}{2l}\right) C_{2(n-k-i-1),2k,2(i+1)}}{B\left(\frac{1}{2l}, \frac{1}{2l}\right)}.
\end{aligned}$$

Moreover, we can see from the expressions of α_i and β_i that the functions $L_{2n-i,i}$ do not affect the first order averaging method and that the constants c, $C_{0,0,2n}$, $\alpha_1, \dots, \alpha_n$, $\beta_1, \dots, \beta_n$ can be chosen arbitrarily.

Now we look for the solution (R, ξ) of the following system of polynomial equations with $R > 0$

$$\begin{aligned}
&\alpha_0 R^{2n-2} + \alpha_1 R^{2n-2}\xi + \alpha_2 R^{2n-4}\xi^3 + \cdots + \alpha_n \xi^{2n-1} = 0, \\
&c\xi^{2n-1} + \beta_1 R^{2n} + \beta_2 R^{2n-2}\xi^2 + \cdots + \beta_n R^2 \xi^{2n-2} + C_{0,0,2n}\xi^{2n} = 0,
\end{aligned}$$

which is equivalent to

$$\alpha_1 R^{2n-2}\xi + \alpha_2 R^{2n-4}\xi^3 + \cdots + \alpha_n \xi^{2n-1} = -\alpha_0 R^{2n-2}, \tag{14}$$

$$\beta_1 R^{2n} + \beta_2 R^{2n-2}\xi^2 + \cdots + \beta_n R^2 \xi^{2n-2} + C_{0,0,2n}\xi^{2n} = -c\xi^{2n-1}. \tag{15}$$

So dividing Eq. (15) by Eq. (14), we have

$$\frac{\beta_1 R^{2n} + \beta_2 R^{2n-2}\xi^2 + \cdots + \beta_n R^2 \xi^{2n-2} + C_{0,0,2n}\xi^{2n}}{\alpha_1 R^{2n-2}\xi + \alpha_2 R^{2n-4}\xi^3 + \cdots + \alpha_n \xi^{2n-1}} = \frac{c\xi^{2n-1}}{\alpha_0 R^{2n-2}},$$

which is equivalent to

$$\frac{\beta_1 + \beta_2\left(\frac{\xi}{R}\right)^2 + \cdots + \beta_n\left(\frac{\xi}{R}\right)^{2n-2} + C_{0,0,2n}\left(\frac{\xi}{R}\right)^{2n}}{\alpha_1\left(\frac{\xi}{R}\right)^2 + \cdots + \alpha_{n-1}\left(\frac{\xi}{R}\right)^{2n-2} + \alpha_n\left(\frac{\xi}{R}\right)^{2n}} = \frac{c}{\alpha_0}\left(\frac{\xi}{R}\right)^{2n-2}.$$

That is

$$\begin{aligned}
&\frac{c\alpha_n}{\alpha_0}\left(\frac{\xi}{R}\right)^{4n-2} + \frac{c\alpha_{n-1}}{\alpha_0}\left(\frac{\xi}{R}\right)^{4n-4} + \cdots + \left(\frac{c\alpha_1}{\alpha_0} - C_{0,0,2n}\right)\left(\frac{\xi}{R}\right)^{2n} \\
&\quad - \beta_n\left(\frac{\xi}{R}\right)^{2n-2} - \cdots - \beta_2\left(\frac{\xi}{R}\right)^2 - \beta_1 = 0.
\end{aligned}$$

Since $c, C_{0,0,2n}, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ can be chosen arbitrarily, we can have just $2n-1$ positive real roots for the variable $(\xi/R)^2$. Note that from Eq. (14), we have

$$\xi = -\frac{\alpha_0}{\alpha_1 + \cdots + \alpha_{n-1}\left(\frac{\xi}{R}\right)^{2n-4} + \alpha_n\left(\frac{\xi}{R}\right)^{2n-2}}.$$

Hence, each positive real solution ξ/R corresponds to a unique solution of the polynomial system (14) and (15). Therefore, by Theorem 3, the differential system (12) can have at most $2n-1$ limit cycles using the averaging method up to first order in ε.

Finally, since the transformation from the original system to system (12) is $r = \varepsilon R$ and $z = \varepsilon\xi$, this means that the at most $2n-1$ limit cycles tend to zero as $\varepsilon \to 0$ in system (2). That is, all the above $2n-1$ limit cycles bifurcate from the origin of system (2). This completes the proof of Theorem 1.

4. Proof of Theorem 2

We assume that

$$P_{2n} = \sum_{i+j+k=2n}^{2n+1} A_{i,j,k} x^i y^j z^k + \text{H.O.T.},$$

$$Q_{2n} = \sum_{i+j+k=2n}^{2n+1} B_{i,j,k} x^i y^j z^k + \text{H.O.T.},$$

$$R_{2n} = \sum_{i+j+k=2n}^{2n+1} C_{i,j,k} x^i y^j z^k + \text{H.O.T.}$$

By the cylinder transformation

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$

system (2) can be changed into

$$\begin{aligned}
\dot r &= \sum_{i=0}^{2n} K_{2n-i,i}(\theta) r^{2n-i} z^i + \sum_{i=0}^{2n+1} K_{2n+1-i,i}(\theta) r^{2n+1-i} z^i + O_{2n+2}(r,z) + \varepsilon\Gamma(\theta) r^{2n-1},\\
\dot\theta &= \frac{1}{r}\left(r^{2n-1} + \sum_{i=0}^{2n} L_{2n-i,i}(\theta) r^{2n-i} z^i + \sum_{i=0}^{2n+1} L_{2n+1-i,i}(\theta) r^{2n+1-i} z^i\right) + O_{2n+1}(r,z),\\
\dot z &= \sum_{i=0}^{2n} M_{2n-i,i}(\theta) r^{2n-i} z^i + \sum_{i=0}^{2n+1} M_{2n+1-i,i}(\theta) r^{2n+1-i} z^i + O_{2n+2}(r,z) + \varepsilon c z^{2n-1}, \quad (16)
\end{aligned}$$

where $\Gamma(\theta)$, $K_{2n-i,i}(\theta)$, $L_{2n-i,i}(\theta)$ and $M_{2n-i,i}(\theta)$ are the same defined in Section 3 and

$$\begin{aligned}
K_{2n+1-i,i}(\theta) &= \sum_{r+s=2n+1-i} \left(A_{r,s,i}\cos^{r+1}\theta\sin^s\theta + B_{r,s,i}\cos^r\theta\sin^{s+1}\theta\right),\\
L_{2n+1-i,i}(\theta) &= \sum_{r+s=2n+1-i} \left(B_{r,s,i}\cos^{r+1}\theta\sin^s\theta - A_{r,s,i}\cos^r\theta\sin^{s+1}\theta\right),\\
M_{2n+1-i,i}(\theta) &= \sum_{r+s=2n+1-i} C_{r,s,i}\cos^r\theta\sin^s\theta.
\end{aligned}$$

By the coordinate transformation $r = \varepsilon R$ and $z = \varepsilon\xi$, system (16) is orbitally equivalent to the following one

$$\frac{dR}{d\theta} = \varepsilon R \frac{\Gamma(\theta)R^{2n-1} + \sum_{i=0}^{2n} K_{2n-i,i}(\theta)R^{2n-i}\xi^i + \varepsilon\sum_{i=0}^{2n+1} K_{2n+1-i,i}(\theta)R^{2n+1-i}\xi^i}{R^{2n-1} + \varepsilon\sum_{i=0}^{2n} L_{2n-i,i}(\theta)R^{2n-i}\xi^i + \varepsilon^2\sum_{i=0}^{2n+1} L_{2n+1-i,i}(\theta)R^{2n+1-i}\xi^i},$$
$$\frac{d\xi}{d\theta} = \varepsilon R \frac{c\xi^{2n-1} + \sum_{i=0}^{2n} M_{2n-i,i}(\theta)R^{2n-i}\xi^i + \varepsilon\sum_{i=0}^{2n+1} M_{2n+1-i,i}(\theta)R^{2n+1-i}\xi^i}{R^{2n-1} + \varepsilon\sum_{i=0}^{2n} L_{2n-i,i}(\theta)R^{2n-i}\xi^i + \varepsilon^2\sum_{i=0}^{2n+1} L_{2n+1-i,i}(\theta)R^{2n+1-i}\xi^i},$$

where we have omitted some terms of order 3 in ε.

Here our purpose is to apply Theorem 4 to system (16) instead of Theorem 3. So take α_i, c, β_i and $C_{0,0,2n}$ defined in Section 3 equal to zero for $i = 1, \dots, n$ and $j = 1, \dots, n$. In this way the functions $F_i^0(R, \xi)$, for $i = 1, 2$, are identically zero, and the averaging theory of first order does not provide any information. Thus we have

$$\frac{dR}{d\theta} = \varepsilon f_1(\theta, R, \xi) + \varepsilon^2\big(g_1(\theta, R, \xi) + h_1(\theta, R, \xi)\big) + \mathrm{O}\big(\varepsilon^3\big),$$
$$\frac{d\xi}{d\theta} = \varepsilon f_2(\theta, R, \xi) + \varepsilon^2\big(g_2(\theta, R, \xi) + h_1(\theta, R, \xi)\big) + \mathrm{O}\big(\varepsilon^3\big),$$

where

$$f_1(\theta, R, \xi) = \Gamma(\theta)R + R^2\sum_{i=0}^{2n} K_{2n-i,i}(\theta)\left(\frac{\xi}{R}\right)^i,$$

$$f_2(\theta, R, \xi) = R^2\sum_{i=0}^{2n-1} M_{2n-i,i}(\theta)\left(\frac{\xi}{R}\right)^i,$$

$$g_1(\theta, R, \xi) = R^3\sum_{i=0}^{2n+1} K_{2n+1-i,i}(\theta)\left(\frac{\xi}{R}\right)^i,$$

$$g_2(\theta, R, \xi) = R^3\sum_{i=0}^{2n+1} M_{2n+1-i,i}(\theta)\left(\frac{\xi}{R}\right)^i,$$

$$K(\theta, R, \xi) = R\sum_{i=0}^{2n} L_{2n-i,i}(\theta)\left(\frac{\xi}{R}\right)^i,$$

and $h_i = -Kf_i$ for $i = 1, 2$. Of course, the 2π-periodic averaging of f_1 and f_2 are zero by our assumptions.

By Proposition 5, the averaged of g_1 is

$$G_1^0 = R^3\left(\lambda_0 + \lambda_1\left(\frac{\xi}{R}\right)^2 + \cdots + \lambda_n\left(\frac{\xi}{R}\right)^{2n}\right),$$

where

$$\lambda_i = \frac{1}{2\pi}\int_0^{2\pi} K_{2(n-i)+1,2i}(\theta)\,d\theta.$$

Similarly the averaged of g_2 is

$$G_2^0 = R^3\left(\omega_0 \frac{\xi}{R} + \omega_1\left(\frac{\xi}{R}\right)^3 + \cdots + \omega_n\left(\frac{\xi}{R}\right)^{2n+1}\right),$$

where

$$\omega_i = \frac{1}{2\pi}\int_0^{2\pi} M_{2(n-i),2i+1}(\theta)\,d\theta.$$

Let $\mathcal{P}_d$ denote a homogeneous polynomial of degree d in $\mathbb{R}[x, y]$. Then we have the following result.

Proposition 6. *If $\int_0^{2\pi} \mathcal{P}_{d_i}(\cos\varphi, \sin\varphi)\,d\varphi = 0$ for $i = 1, 2$ and $d_1 + d_2$ is odd, then*

$$\int_0^{2\pi} \mathcal{P}_{d_1}(\cos\theta, \sin\theta)\left(\int_0^{\theta} \mathcal{P}_{d_2}(\cos\varphi, \sin\varphi)\,d\varphi\right)d\theta = 0.$$

Proof. Since $\int_0^{2\pi} \mathcal{P}_d(\cos\varphi, \sin\varphi)\,d\varphi = 0$, then we know that

$$\int_0^{\theta} \mathcal{P}_d(\cos\varphi, \sin\varphi)\,d\varphi = \sum_{i=1}^{d} \mathcal{Q}_i(\theta) + \text{constant},$$

where $\mathcal{Q}_i(\theta)$ is a trigonometric homogeneous polynomial of degree i in the variables $\cos\theta$ and $\sin\theta$. From the properties 4, 5, 6 and 7, we know that i and d have the same parity. So, by Proposition 5, we get the result. □

We can write h_1 as $h_1 = \mathrm{I}_{h_1} + \mathrm{II}_{h_1}$, where

$$\mathrm{I}_{h_1} = -R^2 \sum_{i=0}^{2n} \Gamma(\theta) L_{2n-i,i}(\theta)\left(\frac{\xi}{R}\right)^i,$$

and by Proposition 6 its averaged is

$$\mathrm{I}_{h_1}^0 = -R^2\left(\eta_0 \frac{\xi}{R} + \eta_1\left(\frac{\xi}{R}\right)^3 + \cdots + \eta_{n-1}\left(\frac{\xi}{R}\right)^{2n-1}\right),$$

where

$$\eta_i = \frac{1}{2\pi}\int_0^{2\pi} \Gamma(\theta) L_{2(n-i)-1,2i+1}(\theta)\,d\theta.$$

Similarly, we have that

$$\mathrm{II}_{h_1} = -R^3\left(\sum_{i=0}^{2n} K_{2n-i,i}(\theta)\left(\frac{\xi}{R}\right)^i\right)\left(\sum_{i=0}^{2n} L_{2n-i,i}(\theta)\left(\frac{\xi}{R}\right)^i\right).$$

By the fact that $\int_0^{2\pi} K_{0,2n}(\theta)L_{0,2n}(\theta)\,d\theta = 0$ and by Proposition 6, its averaged is

$$\mathrm{II}_{h_1}^0 = -R^3\left(\psi_0 + \psi_1\left(\frac{\xi}{R}\right)^2 + \cdots + \psi_{2n-1}\left(\frac{\xi}{R}\right)^{4n-2}\right),$$

where

$$\psi_i = \frac{1}{2\pi}\sum_{l+m=2i}\int_0^{2\pi} K_{2n-l,l}(\theta)L_{2n-m,m}(\theta)\,d\theta.$$

Now we write h_2 as

$$h_2 = -Kf_2 = -R^3\left(\sum_{i=0}^{2n-1} M_{2n-i,i}(\theta)\left(\frac{\xi}{R}\right)^i\right)\left(\sum_{i=0}^{2n} L_{2n-i,i}(\theta)\left(\frac{\xi}{R}\right)^i\right),$$

and its averaged is

$$h_2^0 = -R^3\left(\gamma_0\left(\frac{\xi}{R}\right) + \gamma_1\left(\frac{\xi}{R}\right)^3 + \cdots + \gamma_{2n-1}\left(\frac{\xi}{R}\right)^{4n-1}\right),$$

where

$$\gamma_i = \frac{1}{2\pi}\sum_{l+m=2i+1}\int_0^{2\pi} M_{2n-l,l}(\theta)L_{2n-m,m}(\theta)\,d\theta.$$

By Theorem 4, we need to calculate the averaged of the part generated by the first order terms of ε. So Eq. (9) here can be written as

$$F^1(t,R,\xi) = Df(R,\xi)\cdot\int_0^t f(s,R,\xi)\,ds,$$

where Df is the Jacobian matrix with respect to the variable (R,ξ), and its elements are

$$\frac{\partial f_1}{\partial R} = \Gamma(\theta) + R\sum_{i=0}^{2n}(2-i)K_{2n-i,i}(\theta)\left(\frac{\xi}{R}\right)^i,$$

$$\frac{\partial f_1}{\partial \xi} = R\sum_{i=1}^{2n} iK_{2n-i,i}(\theta)\left(\frac{\xi}{R}\right)^{i-1},$$

$$\frac{\partial f_2}{\partial R} = R\sum_{i=0}^{2n-1}(2-i)M_{2n-i,i}(\theta)\left(\frac{\xi}{R}\right)^i,$$

$$\frac{\partial f_2}{\partial \xi} = R\sum_{i=1}^{2n-1} iM_{2n-i,i}\left(\frac{\xi}{R}\right)^{i-1}.$$

Thus,

$$F_1^1 = \frac{\partial f_1}{\partial R}\int_0^\theta f_1(\varphi,R,\xi)\,d\varphi + \frac{\partial f_1}{\partial \xi}\int_0^\theta f_2(\varphi,R,\xi)\,d\varphi = \mathrm{I}_{y_1} + \mathrm{II}_{y_1}.$$

Then the averaged function of I_{y_1} is

$$\mathrm{I}_{y_1}^0 = R^2\left(\mu_1\left(\frac{\xi}{R}\right)^3 + \mu_2\left(\frac{\xi}{R}\right)^5 + \cdots + \mu_{n-1}\left(\frac{\xi}{R}\right)^{2n-1}\right) + R^3\left(\nu_1\left(\frac{\xi}{R}\right)^2 + \nu_2\left(\frac{\xi}{R}\right)^4 + \cdots + \nu_{2n-1}\left(\frac{\xi}{R}\right)^{4n-2}\right),$$

with

$$\mu_i = \frac{i}{\pi}\int_0^{2\pi} \Gamma(\theta)\left(\int_0^{\theta} K_{2(n-i)-1,2i+1}(\varphi)\,d\varphi\right)d\theta,$$

$$\nu_i = \frac{1}{2\pi}\sum_{l+m=2i}\int_0^{2\pi}(2-l)K_{2n-l,l}(\theta)\left(\int_0^{\theta} K_{2m-m,m}(\varphi)\,d\varphi\right)d\theta.$$

Moreover

$$\mathrm{II}_{y_1}^0 = R^3\left(\chi_0 + \chi_1\left(\frac{\xi}{R}\right)^2 + \cdots + \chi_{2n-1}\left(\frac{\xi}{R}\right)^{4n-2}\right),$$

where

$$\chi_i = \frac{1}{2\pi}\sum_{l+m=2i+1}\int_0^{2\pi} l K_{2n-l,l}(\theta)\left(\int_0^{\theta} M_{2n-m,m}(\varphi)\,d\varphi\right)d\theta.$$

Similarly, we have

$$y_2 = \frac{\partial f_2}{\partial R}\int_0^{\theta} f_1(\varphi, R, \xi)\,d\varphi + \frac{\partial f_2}{\partial \xi}\int_0^{\theta} f_2(\varphi, R, \xi)\,d\varphi = \mathrm{I}_{y_2} + \mathrm{II}_{y_2},$$

where

$$\mathrm{I}_{y_2}^0 = R^2\left(\tau_0 + \tau_1\left(\frac{\xi}{R}\right)^2 + \cdots + \tau_{n-1}\left(\frac{\xi}{R}\right)^{2n-2}\right) + R^3\left(\varsigma_0\left(\frac{\xi}{R}\right) + \varsigma_1\left(\frac{\xi}{R}\right)^3 + \cdots + \varsigma_{2n-1}\left(\frac{\xi}{R}\right)^{4n-1}\right)$$

with

$$\tau_i = \frac{1-i}{\pi}\int_0^{2\pi} \Gamma(\theta)\left(\int_0^{\theta} M_{2(n-i),2i}(\varphi)\,d\varphi\right)d\theta,$$

$$\varsigma_i = \frac{1}{2\pi}\sum_{l+m=2i+1}\int_0^{2\pi}(2-l)M_{2n-l,l}(\theta)\left(\int_0^{\theta} K_{2n-m,m}(\varphi)\,d\varphi\right)d\theta.$$

In particular, we have $\tau_1 = 0$, and

$$\mathrm{II}_{y_2}^0 = R^3\left(\sigma_0\frac{\xi}{R} + \sigma_1\left(\frac{\xi}{R}\right)^3 + \cdots + \sigma_{2n-3}\left(\frac{\xi}{R}\right)^{4n-5}\right),$$

where

$$\sigma_i = \frac{1}{2\pi} \sum_{l+m=2i+2} \int_0^{2\pi} l M_{2n-l,l}(\theta) \left(\int_0^{\theta} M_{2n-m,m}(\varphi)\, d\varphi \right) d\theta.$$

By Theorem 4, we must study the solutions of the polynomial system

$$R^2 \sum_{i=0}^{n-1} \mathcal{A}_i \left(\frac{\xi}{R}\right)^{2i+1} = -R^3 \sum_{i=0}^{2n-1} \mathcal{B}_i \left(\frac{\xi}{R}\right)^{2i},$$
$$R^2 \sum_{i=0}^{n-1} \mathcal{C}_i \left(\frac{\xi}{R}\right)^{2i} = -R^3 \sum_{i=0}^{2n-1} \mathcal{D}_i \left(\frac{\xi}{R}\right)^{2i+1},$$

where

$$\begin{aligned}
&\mathcal{A}_0 = -\eta_0, \qquad \mathcal{A}_i = \mu_i - \eta_i, \quad i = 1, \ldots, n-1, \\
&\mathcal{B}_0 = \lambda_0 - \psi_0 + \chi_0, \qquad \mathcal{B}_i = \lambda_i - \psi_i + \chi_i + \nu_i, \quad i = 1, \ldots, n, \\
&\mathcal{B}_j = -\psi_j + \chi_j + \nu_j, \quad j = n+1, \ldots, 2n-1, \\
&\mathcal{C}_i = \tau_i, \quad i = 0, \ldots, n-1, \\
&\mathcal{D}_i = \omega_i - \gamma_i + \varsigma_i + \sigma_i, \quad i = 0, \ldots, n, \\
&\mathcal{D}_j = -\gamma_j + \varsigma_j + \sigma_j, \quad j = n+1, \ldots, 2n-2, \\
&\mathcal{D}_{2n-1} = -\gamma_{2n-1} + \varsigma_{2n-1}.
\end{aligned}$$

Dividing the first equation by the second one, we have

$$\frac{\mathcal{A}_0 \frac{\xi}{R} + \mathcal{A}_1 \left(\frac{\xi}{R}\right)^3 + \cdots + \mathcal{A}_{n-1} \left(\frac{\xi}{R}\right)^{2n-1}}{\mathcal{C}_0 + \mathcal{C}_1 \left(\frac{\xi}{R}\right)^2 + \cdots + \mathcal{C}_{n-1} \left(\frac{\xi}{R}\right)^{2n-2}} = \frac{\mathcal{B}_0 + \mathcal{B}_1 \left(\frac{\xi}{R}\right)^2 + \cdots + \mathcal{B}_{2n-1} \left(\frac{\xi}{R}\right)^{4n-2}}{\mathcal{D}_0 \frac{\xi}{R} + \mathcal{D}_1 \left(\frac{\xi}{R}\right)^3 + \cdots + \mathcal{D}_{2n-1} \left(\frac{\xi}{R}\right)^{4n-1}},$$

which is equivalent to the following polynomial equation of degree $3n-1$ in the variable $z = (\xi/R)^2$

$$\begin{aligned}
&z\left(\mathcal{A}_{n-1} z^{n-1} + \cdots + \mathcal{A}_1 z + \mathcal{A}_0\right)\left(\mathcal{D}_{2n-1} z^{2n-1} + \cdots + \mathcal{D}_1 z + \mathcal{D}_0\right) \\
&\quad - \left(\mathcal{B}_{2n-1} z^{2n-1} + \cdots + \mathcal{B}_1 z + \mathcal{B}_0\right)\left(\mathcal{C}_{n-1} z^{n-1} + \cdots + \mathcal{C}_1 z + \mathcal{C}_0\right) = 0.
\end{aligned} \tag{17}$$

So it can have $3n-1$ positive real roots at most, moreover, each root with respect to the variable $(\xi/R)^2$ provides a unique solution in the variables (R, ξ) using the equality

$$\xi = -\frac{\mathcal{D}_{2n-1} \left(\frac{\xi}{R}\right)^{4n-2} + \cdots + \mathcal{D}_1 \left(\frac{\xi}{R}\right)^2 + \mathcal{D}_0}{\mathcal{C}_{n-1} \left(\frac{\xi}{R}\right)^{2n-2} + \cdots + \mathcal{C}_1 \left(\frac{\xi}{R}\right)^2 + \mathcal{C}_0}.$$

Now we claim that by choosing conveniently the coefficients of system (2), we can have $3n-1$ roots of (17). To prove this, we note that the coefficients $\mathcal{B}_i$ and $\mathcal{D}_i$ can be chosen arbitrary for $i = 0, \ldots, n$, because λ_i and ω_i can be chosen arbitrary. Then, let

$$\begin{aligned}
&M_{2n,0} = C_{1,2n-1,0} \cos\theta \sin^{2n-1}\theta, \\
&M_{2n-2k-2,2k+2} = 0, \quad k = 0, \ldots, n-1, \\
&\Gamma = \left(-\frac{1}{2n-3} \cos^{2n-2}\theta + \cos^{2n-4}\theta \sin^2\theta\right) p_3.
\end{aligned}$$

Thus,

$$\mathcal{C}_0 = \frac{B((2n-3)/2, (2n+1)/2) p_3 C_{1,2n-1,0}}{(2n-1)\pi}, \qquad \mathcal{C}_i = 0, \quad i = 1, \dots, n.$$

So the coefficients of the monomials of degree from 0 to $n-1$ can be chosen arbitrarily in Eq. (17) . Therefore, if $L_{1,2n-1} = -A_{0,1,2n-1} \sin^2\theta$, then $K_{1,2n-1} = A_{0,1,2n-1} \cos\theta \sin\theta$, which implies

$$\mathcal{A}_{n-1} = \frac{1}{\pi} B\big((2n-3)/2, 3/2\big) p_3 A_{0,1,2n-1}.$$

So the coefficients of the monomials of degree from n to $2n$ can be chosen arbitrarily in Eq. (17). Finally, if we take

$$M_{2n-2i-1,2i+1} = \sin^{2n-2i-1}\theta, \quad i = 0, \dots, n-1,$$
$$L_{2n-2j,2j} = -A_{0,2n-2j,2j} \sin^{2n-2j+1}\theta, \quad j = 0, \dots, n,$$

then

$$K_{2n-2j,2j} = A_{0,2n-2j,2j} \cos\theta \sin^{2n-2j}\theta.$$

So we have

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi} M_{2n-1,1}(\theta)\left(\int_0^{\theta} K_{2n-2j,2j}(\varphi)\,d\varphi - L_{2n-2j,2j}(\theta)\right)d\theta \\ &\quad = \frac{n-j+1}{(2n-2j+1)\pi} B\big((4n-2j+1)/2, 1/2\big) A_{0,2n-2j,2j}.\end{aligned} \tag{18}$$

That is, let S be a set of variables and if we denote $L(S)$ the linear combination of all variables in S, then from the expressions of γ_j, ς_j and σ_j, we have

$$\mathcal{D}_j = L(A_{0,2n-2k,2k}, k = 0, \dots, 2n-j-1) + \text{constant}_j \tag{19}$$

for $j = n+1, \dots, 2n-1$. More precisely, let $Y = (\mathcal{D}_{n+1}, \dots, \mathcal{D}_{2n-1})^T$ and $X = (A_{0,4,2n-4}, \dots, A_{0,2n,0})^T$, we can write system (19) in a matrix form $Y = SX + C$ and the coefficient matrix S is upper-triangular with non-zero elements in the main diagonal by Eq. (18). So the coefficients $\mathcal{D}_j$ can be chosen arbitrarily for $j = n+1, \dots, 2n-1$, which means that the coefficients of the monomials with degree from $2n+1$ to $3n-1$ in Eq. (17) can also be chosen arbitrarily. By the same arguments in Section 3, we get the result.

References

[1] N.G. Lloyd, The number of limit cycles of polynomial systems in the plane, Bull. Inst. Math. Appl. 24 (11–12) (1988) 161–365.

[2] N.G. Lloyd, Small Amplitude Limit Cycles of Polynomial Differential Equations, Lecture Notes in Math., vol. 1032, Springer-Verlag, Berlin, 1983.

[3] A. Buică, J. Llibre, Averaging methods for finding periodic orbits via Brouwer degree, Bull. Sci. Math. 128 (2004) 7–22.

[4] J. Llibre, Averaging theory and limit cycles for quadratic systems, Rad. Mat. 11 (2002) 1–14.

[5] J.A. Sanders, F. Verhulst, Averaging Methods in Nonlinear Dynamical Systems, Applied Mathematical Sci., vol. 59, Springer-Verlag, New York, 1985.

[6] F. Verhulst, Nonlinear Differential Equations and Dynamical Systems, Universitext, Springer-Verlag, Berlin, 1996.